

On Convergent Interpolatory Polynomials

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Let

$$X_n: -1 \leq x_{nn} < x_{n-1,n} < \cdots < x_{1n} \leq 1 \quad (n = 1, 2, \dots) \quad (1)$$

be a system of nodes of interpolation. We are interested in finding necessary and sufficient conditions on (1) in order that for every $f(x) \in C[-1, 1]$ and $\varepsilon > 0$ there exist polynomials $p_n(x) \in \Pi_{[n(1+\varepsilon)]}$ such that

$$p_n(x_{kn}) = f(x_{kn}) \quad (k = 1, \dots, n; n = 1, 2, \dots) \quad (2)$$

and

$$\lim_{n \rightarrow \infty} \|f(x) - p_n(x)\| = 0. \quad (3)$$

Here Π_m is the set of algebraic polynomials of degree at most m , $C[-1, 1]$ is the space of continuous functions on the interval $[-1, 1]$, and $\|\cdot\|$ is the maximum (over $[-1, 1]$) norm.

Let $x_{kn} = \cos t_{kn}$, $0 \leq t_{1n} < t_{2n} < \cdots < t_{nn} \leq \pi$, and for an arbitrary interval $I \subseteq [0, \pi]$, denote

$$N_n(I) = \sum_{t_{kn} \in I} 1.$$

In this paper we shall prove the following

THEOREM. *For every $f(x) \in C[-1, 1]$ and $\varepsilon > 0$ there exists a sequence of polynomials $p_n(x) \in \Pi_{[n(1+\varepsilon)]}$ such that (2) and*

$$\|f(x) - p_n(x)\| = O(E_{[n(1+\varepsilon)]}(f)) \quad (4)$$

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hold, if and only if

$$\overline{\lim}_{n \rightarrow \infty} \frac{N_n(I_n)}{n|I_n|} \leq \frac{1}{\pi} \quad \text{whenever} \quad \lim_{n \rightarrow \infty} n|I_n| = \infty \quad (|I_n| = \text{length of } I_n) \quad (5)$$

and

$$\lim_{n \rightarrow \infty} \min_{1 \leq i \leq n-1} n(t_{i+1,n} - t_{i,n}) > 0. \quad (6)$$

Here the O sign refers to $n \rightarrow \infty$ and indicates a constant depending only on ε ; $E_m(f)$ is the best uniform approximation of $f(x)$ by polynomials of degree at most n .

This theorem, in a slightly weaker form ((4) replaced by (3)) was stated in [1, Theorem 4]. There was no proof given, only an indication that it is a simple modification of the proof of Theorem 3. While we were unable to reconstruct this "simple modification" (it was probably not that simple at all), we found a proof which we think worthwhile to publish, since the above theorem is a fundamental and frequently quoted result of the theory of interpolation.

The proof is long and sophisticated, and in order to make it more understandable we break it into a series of lemmas. First we aim at the sufficiency of conditions (5)–(6).

LEMMA 1. Under conditions (5), (6) for any $\varepsilon > 0$ there exists a system of nodes (in not necessarily decreasing order)

$$\begin{aligned} Y_n : y_k &= y_{kn} = \cos \eta_k, \\ \eta_k &= \eta_{km} = \frac{2k-1+d_k}{m} \frac{\pi}{2}, \\ k &= 1, \dots, m = [n(1+\varepsilon)]; \quad n \geq n_0 \end{aligned} \quad (7)$$

such that

- (a) the x_i 's are among the y_k 's;
- (b) $n(\eta_{k+1} - \eta_k) \geq c > 0$ ($k = 1, \dots, m$; $n \geq n_0$) with an absolute constant c , and
- (c) $|\sum_{k=1}^s d_k| \leq A$ ($s = 1, \dots, m$) with a constant $A = A(\varepsilon)$.

Proof. Condition (5) implies that for any $\varepsilon > 0$, there exist $\Delta(\varepsilon)$ and $n_0(\varepsilon)$ such that

$$\frac{N_n(I)}{n|I|} \leq \frac{1}{\pi} + \varepsilon \quad \text{whenever} \quad n(I) \geq \Delta(\varepsilon) \quad \text{and} \quad n \geq n_0(\varepsilon). \quad (8)$$

Let

$$\Delta = \max \left(\Delta \left(\frac{\varepsilon}{4} \right), \frac{30}{\varepsilon} \right)$$

and consider the intervals

$$J_i = \left[\frac{i\Delta}{n}, \frac{(i+1)\Delta}{n} \right) \quad \left(i = 0, \dots, \left[\frac{\pi n}{\Delta} \right] - 1 \right).$$

By (8) and $n|J_i| = \Delta$,

$$N_n(J_i) \leq \left(\frac{1}{\pi} + \frac{\varepsilon}{4} \right) \Delta \quad \left(i = 0, \dots, \left[\frac{\pi n}{\Delta} \right] - 1 \right).$$

The number of equidistant nodes

$$\theta_k = \frac{2k-1}{m+1} \frac{\pi}{2} \quad (k = 1, \dots, m+1)$$

in J_i is $\geq (\Delta(m+1))/\pi n > (\Delta/\pi)(1+\varepsilon)$, i.e., at least $\Delta\varepsilon(1/\pi - 1/4) > 3$ more than $N_n(J_i)$.

We shall construct the η_k 's in two phases. In the first phase, in each J_i where at least one t_k occurs, replace the θ_j 's by these t_k 's, and leave the remaining θ_j 's unchanged. According to the previous argument, there is at least one such unchanged θ_j in each J_i (call them free nodes). This system fulfils so far only (a). We would like to ensure (b). By (6) we may assume that

$$t_{i+1} - t_i \geq \frac{c}{n} \quad (c < 1, i = 1, \dots, n-1). \quad (9)$$

Consider those remaining θ_j 's for which there exists a t_i such that

$$0 < |\theta_j - t_i| \leq \frac{c}{7n}, \quad (10)$$

and move these θ_j 's alternatively to the left or to the right with a distance $2c/(7n)$. Then these translated θ_k 's will be farther than $c/(7n)$ from any t_i

(see (9)), and the distance of adjacent new θ_j 's will be at least $\pi/(m+1) - 4c/(7n) > (\pi/2 - 4/7)(1/n)$. Thus the change in the contribution of the d_k 's will be $O(1)$, and (b) is satisfied. After completing these steps, at least one free node remains in each J_i .

In the second phase we want to ensure (c) by further modifications. Divide consecutive J_i 's into groups of 10Δ members. In each J_i , the maximal contribution of d_k 's is $< (1/\pi + \varepsilon/4)\Delta \cdot 2(1+\varepsilon)\Delta/\pi < \Delta^2$ (we may assume that $\varepsilon < 1$); thus for the whole group it is $< 10\Delta^3$. We would like to arrive at a situation where the *total* contribution of d_k 's at the end of each group is $< 10\Delta^3$. We proceed by induction on the number of groups. As we have seen, in the first group the contribution is $< 10\Delta^3$. Assume that the total contribution of the first $a-1$ groups is $< 10\Delta^3$, and, without loss of generality we may assume that this contribution is nonnegative. By proper changes, we would like to have a contribution in the a th group between $-10\Delta^3$ and 0, thus ensuring a total contribution in the first a groups between $-10\Delta^3$ and $10\Delta^3$. In the a th group, the total contribution is between $-10\Delta^3$ and $10\Delta^3$. If it is negative, we are done. Thus assume that it is between 0 and $10\Delta^3$, and omit a free node from the $(5\Delta+2)$ nd J_i and replace it by the midpoint of any two adjacent nodes in the $(5\Delta-2)$ nd J_i . The result is a decrease of at least $2 \cdot 2(1+\varepsilon)\Delta/\pi$ and at most $4 \cdot 2(1+\varepsilon)\Delta/\pi$ in the contribution of the d_k 's in the a th group. If this change transforms this contribution below zero, then we are done. If not, then omit a free node from the $(5\Delta+3)$ rd J_i and replace it by the midpoint of any two adjacent nodes in the $(5\Delta-3)$ rd J_i . The result is another decrease of at least $4 \cdot 2(1+\varepsilon)\Delta/\pi$ and at most $6 \cdot 2(1+\varepsilon)\Delta/\pi$ in the contribution of the d_k 's in the a th group. If this second change transforms this contribution below zero, then we are done; otherwise continue this procedure with the $(5\Delta+4)$ th and $(5\Delta-4)$ th J_i 's, etc. Before exhausting all the possibilities we must arrive at the desired situation, because the decrease of the contribution in the a th group after all the possible changes would be at least

$$(2+4+\cdots+10\Delta-2)(1+\varepsilon)\Delta/\pi > \frac{2\Delta}{\pi} 5\Delta(5\Delta-1) > \frac{40\Delta^3}{\pi}$$

which is greater than $10\Delta^3$, the original maximal contribution in the a th group. (Even if we needed the last change here, its maximal contribution is $< 10\Delta \cdot 2(1+\varepsilon)\Delta/\pi < 13\Delta^2 < 10\Delta^3$, so we never get under $-10\Delta^3$.)

After making all these changes in each group, we arrive at a situation where the total contribution of the d_k 's at the last J_i in a group will be $< 10\Delta^3$. But it is clear from the previous argument that $|d_k| < 13\Delta^2$, and since the number of d_k 's in a group is $< 10\Delta \cdot (\Delta(1+\varepsilon)/\pi) + 5\Delta < 12\Delta^2$, the

contribution *inside* a group cannot be higher than $13\Delta^2 \cdot 12\Delta^2$, i.e., bounded again. Thus Lemma 1 is completely proved. ■

LEMMA 2. *For the fundamental functions of Lagrange interpolation based on the nodes (7) we have*

$$\|I_j(Y_m, x)\| = O(1) \quad (k = 1, \dots, m).$$

Proof. Let

$$\begin{aligned} Z_m : z_k &= \cos \frac{2k-1}{2m} \pi \quad (k = 1, \dots, m); \\ T_m(x) &= \prod_{k=1}^m (x - z_k), \\ \Omega_m(x) &= \prod_{k=1}^m (x - y_k). \end{aligned} \tag{11}$$

Then for a fixed k , the number v_k of y_i 's for which $\text{sgn}(y_k - y_i) = \text{sgn}(k - i)$ is evidently $v_k = o(1)$, and thus denoting $A_k = \{i \mid \text{sgn}(y_k - y_i) = \text{sgn}(k - i)\}$, $B_k = \{1, \dots, m\} \setminus A_k$ we have

$$\begin{aligned} \left| \frac{T'_m(z_k)}{\Omega'(y_k)} \right| &= \prod_{i \in A_k} \frac{z_k - z_i}{y_i - y_k} \prod_{i \in B_k} \frac{z_k - z_i}{y_k - y_i} \\ &= O(1) \prod_{i \in B_k} \left(1 + \frac{z_k - y_k + y_i - z_i}{y_k - y_i} \right) \\ &= O(1) \exp \sum_{i \in B_k} \frac{z_k - y_k + y_i - z_i}{y_k - y_i} \\ &= O(1) \exp \sum_{i \neq k} \frac{z_k - y_k + y_i - z_i}{y_k - y_i}. \end{aligned}$$

Here, using $|d_k| = O(1)$ (see Lemma 1(c)), we get for $1 \leq k \leq m/2$

$$\begin{aligned} |z_k - y_k| \sum_{i \neq k} \frac{1}{y_k - y_i} &= O\left(\frac{k|d_k|}{m^2}\right) \left\{ \left| \sum_{i \neq k} \frac{1}{z_i - z_k} \right| + \sum_{i \neq k} \left| \frac{y_k - z_k + z_i - y_i}{(z_k - z_i)(y_k - y_i)} \right| \right\} \\ &= O\left(\frac{k}{m^2}\right) \left\{ \left| \frac{T''_m(z_k)}{T'_m(z_k)} \right| + \sum_{i \neq k} \frac{(k|d_k|/m^2) + (i|d_i|/m^2)}{((k-i)^2 \min(k+i, m/2)^2)/m^4} \right\} \\ &= O\left(\frac{k}{m^2}\right) \left\{ \frac{m^2}{k^2} + \frac{m^2}{k} \sum_{i \neq k} \frac{1}{(k-i)^2} \right\} = O(1), \end{aligned}$$

and using Abel's transform

$$\begin{aligned}
 \left| \sum_{i \neq k} \frac{z_i - y_i}{y_k - y_i} \right| &= \left| \sum_{i \neq k} \frac{2 \sin(d_i \pi / 4m) \sin((4i - 2 + d_i) / 4m) \pi}{y_k - y_i} \right| \\
 &= \left| \sum_{i \neq k} \frac{(d_i \pi / 2m) \sin((4i - 2 + d_i) / 4m) \pi + O(m^{-3})}{y_k - y_i} \right| \\
 &= O \left\{ \frac{1}{m} \sum_{i \neq k, k+1} \left(\frac{\sin((4i + 2 + d_i) / 4m) \pi}{y_k - y_{i+1}} \right. \right. \\
 &\quad \left. \left. - \frac{\sin((4i - 2 + d_i) / 4m) \pi}{y_k - y_i} \right) \sum_{j=1}^i d_j \right\} + O(1) \\
 &= O \left(\frac{1}{m} \right) \cdot \sum_{i \neq k, k+1} \left(\frac{(i/m) |y_i - y_{i+1}|}{|y_k - y_{i+1}| \cdot |y_k - y_i|} + \frac{i/m^2}{|y_k - y_i|} \right) + O(1) \\
 &= O \left(\frac{1}{m} \right) \sum_{i \neq k} \left(\frac{i^2/m^3}{(k^2 - i^2)^2/m^4} + \frac{i/m^2}{|k^2 - i^2|/m^2} \right) + O(1) \\
 &= O \left(\sum_{i \neq k} \frac{1}{(k-i)^2} + \frac{1}{m} \sum_{i \neq k} \frac{1}{|k-i|} + 1 \right) = O(1),
 \end{aligned}$$

and similarly for $m/2 \leq k \leq m$. Hence

$$|T'_m(z_k)| = O(|\Omega'_m(y_k)|) \quad (k = 1, \dots, m). \quad (12)$$

Now let $|x| \leq 1$ be arbitrary and $0 \leq j \leq m$ be such that $z_{j+1} \leq x \leq z_j$ (we take $z_0 = 1$ and $z_{m+1} = -1$). Then similarly as before, denoting $u \in (z_{j+1}, z_j)$ for which $T_m(u)$ is a local maximum, the number $v(x)$ of i 's for which $\operatorname{sgn}((x - y_i)/(u - z_i)) = -1$ is evidently $v(x) = O(1)$. Hence

$$\begin{aligned}
 \left| \prod_{i=k} \frac{x - y_i}{u - z_i} \right| &= \prod_{\operatorname{sgn}((x - y_i)/(u - z_i)) = -1} \left| \frac{x - y_i}{u - z_i} \right| \\
 &\quad \times \prod_{\operatorname{sgn}((x - y_i)/(u - z_i)) \geq 0} \left(1 + \frac{x - u + z_i - y_i}{u - z_i} \right) \\
 &= O(1) \exp \sum_{\operatorname{sgn}((x - y_i)/(u - z_i)) \geq 0} \frac{x - u + z_i - y_i}{u - z_i} \\
 &= O(1) \exp \left\{ |x - u| \left(\left| \frac{T'_m(u)}{T_m(u)} \right| + \sum_{\operatorname{sgn}((x - y_i)/(u - z_i)) = -1} \frac{1}{|u - z_i|} \right) \right. \\
 &\quad \left. + \sum_{i \neq j} \frac{i/m^2}{|j^2 - i^2|/m^2} \right\} \\
 &= O(1) \exp O \left\{ \frac{j}{m^2} \left(\frac{m^2}{j} + v(x) \cdot \frac{m^2}{j} \right) + \frac{1}{j} \right\} = O(1).
 \end{aligned}$$

Thus using (12) we get

$$\left| \frac{l_k(Y_m, x)}{l_k(Z_m, u)} \right| = \left| \frac{T'_m(z_k)}{\Omega'_m(y_k)} \prod_{i \neq k} \frac{x - y_i}{u - z_i} \right| = O(1) \quad (k = 1, \dots, m);$$

i.e., using Fejér's result $\|l_k(z_m, u)\| \leq \sqrt{2}$ ($k = 1, \dots, m$) we get the statement of the lemma. ■

After these preliminaries, the sufficiency of conditions (5), (6) is easily proved. Let $s = [n\varepsilon/3]$, and apply Lemma 1 with $\varepsilon/3$ instead of ε ; then $m = [n(1 + \varepsilon/3)]$. Let $g(x) \in \Pi_{[n(1 + \varepsilon)]}$ be the best approximating polynomial of $f(x)$. Consider

$$p_n(x) = q(x) + \sum_{j=0}^s \left\{ \sum_{z_{j+1}, s < y_k \leq z_{j,s}} \frac{(f(x_k) - q(x_k)) l_k(Y_m, x)}{\{l_j(Z_s, y_k) + l_{j+1}(Z_s, y_k)\}^2} \right. \\ \left. \times \{l_j(Z_s, x) + l_{j+1}(Z_s, x)\}^2 \right\}.$$

Since by the well-known Erdős-Turán result [2, Lemma IV]

$$l_j(Z_s, y_k) + l_{j+1}(Z_s, y_k) \geq 1 \quad (z_{j+1} < y_k \leq z_j), \quad (13)$$

the definition of $p_n(x)$ makes sense. Now

$$\deg p_n \leq m - 1 + 2(s - 1) < n \left(1 + \frac{\varepsilon}{3} \right) + \frac{2n\varepsilon}{3} = n(1 + \varepsilon),$$

and evidently

$$p_n(y_i) = f(y_i) \quad (i = 1, \dots, m).$$

This proves (2), since by Lemma 1(a) the x_k 's are among the y_i 's. By the definition of $q(x)$, (13), Lemma 2, and the inequality $(a + b)^2 \leq 2(a^2 + b^2)$ we get

$$\|f(x) - p_n(x)\| \leq \|f(x) - q(x)\| \left\{ 1 + O \left[\left\| \sum_{j=0}^s l_j(Z_s, x)^2 \sum_{z_{j+1} < y_k \leq z_j} 1 \right\| \right] \right\} \\ = O(E_{[n(1 + \varepsilon)]}(f)) \left\| \sum_{j=0}^s l_j(Z_s, x)^2 \right\|,$$

since by Lemma 1(b), $\sum_{z_{j+1} < y_k \leq z_j} 1 = O(1)$. But here again by Fejér's result

$$\left\| \sum_{j=0}^s l_j(Z_s, x)^2 \right\| \leq 2$$

and thus (4) is also proved.

To prove the necessity of (6), assume that there exists a sequence $i_1 < i_2 < \dots$ such that

$$\lim_{n \rightarrow \infty} n(t_{i_n+1,n} - t_{i_n,n}) = 0.$$

Hence passing to monotone subsequences (if necessary), there exists a $t \in [0, \pi]$ such that

$$\lim_{n \rightarrow \infty} t_{i_n,n} = t, \quad t_{i_n+1,n} - t_{i_n,n} \leq \frac{\varepsilon_n}{n}, \quad \lim_{n \rightarrow \infty} \varepsilon_n = 0, \quad (14)$$

and the sequences $\{t_{i_n,n}\}$ and $\{t_{i_n+1,n}\}$ have no points in common. Also, we may assume that at least one of these sequences, say $\{t_{i_n,n}\}$, is strictly monotone. Then define

$$f(t_{i_n,n}) = 0, \quad f(t_{i_n+1,n}) = \sqrt{\varepsilon_n},$$

and f is continuous and linear between these nodes. Because of (14), this defines an $f(x) \in C[-1, 1]$. By (2) and the Bernstein inequality

$$\begin{aligned} \frac{n}{\sqrt{\varepsilon_n}} &\leq \frac{f(\cos t_{i_n+1,n}) - f(\cos t_{i_n,n})}{t_{i_n+1,n} - t_{i_n,n}} \\ &= \frac{p_n(\cos t_{i_n+1,n}) - p_n(\cos t_{i_n,n})}{t_{i_n+1,n} - t_{i_n,n}} \\ &= \frac{d}{dt} p_n(\cos t) \Big|_{t=\xi} = O(n) \|p_n\| \quad (\xi \in (t_{i_n,n}, t_{i_n+1,n})), \end{aligned}$$

i.e., $\|p_n\| \geq 1/\sqrt{\varepsilon_n} \rightarrow \infty$ as $n \rightarrow \infty$, which shows that (4) cannot hold. Hence (6) is necessary.

The proof of the necessity of (5) is more difficult. First we prove the following.

LEMMA 3. Let $I_n \subset [-\pi, \pi]$ ($n \in \mathbb{N}$) and let t_n be a sequence of trigonometric polynomials of order at most r_n such that $r_n |I_n| \rightarrow \infty$ and $\|t_n\| \leq M$ ($n \in \mathbb{N}$) ($r_n \uparrow \infty$). Denote by $Q(I_n)$ the number of $+1, -1, +1, \dots$ oscillations of t_n on I_n . Then

$$\lim_{n \rightarrow \infty} \frac{Q(I_n)}{r_n |I_n|} \leq \frac{1}{\pi}.$$

Proof. Assume to the contrary that $Q(I_n)/r_n |I_n| > (1 + \delta)/\pi$ for some $\delta > 0$ and $n \in \Omega$ ($\Omega \subset \mathbb{N}$ infinite), where we may take $I_n(-a_n, a_n)$ and $0 < a_n < \pi - 2\delta_1$. Let now s_n be an even integer such that

$\sqrt{r_n a_n} < s_n < 2\sqrt{r_n a_n}$ and let $\varepsilon_n = \pi M a_n / (s_n \sin \delta_1)$. Consider the trigonometric polynomial

$$u_n(x) = t_n(x) + \frac{1}{2} \left(\frac{\sin(x/2)}{\sin(a_n/2)} \right)^{s_n} \cos \left(r_n - \frac{s_n}{2} \right) x.$$

of order at most r_n . Evidently, on $[-a_n, a_n]$, u_n has at least $Q(I_n) - 1$ zeros. If $x \notin (-a_n - \varepsilon_n, a_n + \varepsilon_n)$ we have for s_n large enough

$$\begin{aligned} \left(\frac{\sin(x/2)}{\sin(a_n/2)} \right)^{s_n} &\geq \left(\frac{\sin((a_n + \varepsilon_n)/2)}{\sin(a_n/2)} \right)^{s_n} \\ &= \left(1 + \frac{2 \sin(\varepsilon_n/4) \cos(a_n/2 + \varepsilon_n/4)}{\sin(a_n/2)} \right)^{s_n} \\ &\geq \left(1 + \frac{2\varepsilon_n \sin \delta_1}{\pi a_n} \right)^{s_n} \\ &= \left(1 + \frac{2M}{s_n} \right)^{s_n} \geq 2^{2M} > 2M. \end{aligned}$$

Thus u_n has at least $(2\pi - 2a_n - 2\varepsilon_n)((2r_n - s_n)/2\pi) - 4$ zeros in $[-\pi, \pi] \setminus (-a_n - \varepsilon_n, a_n + \varepsilon_n)$. Therefore

$$Q(I_n) + (2\pi - 2a_n - 2\varepsilon_n) \frac{2r_n - s_n}{2\pi} \leq 2r_n + 5,$$

i.e.,

$$\begin{aligned} Q(I_n) &\leq 5 + \frac{2a_n r_n}{\pi} + \frac{2\varepsilon_n r_n}{\pi} + s_n, \\ \frac{1 + \delta}{\pi} &< \frac{Q(I_n)}{r_n |I_n|} = \frac{Q(I_n)}{2r_n a_n} \leq \frac{1}{\pi} + c \left(\frac{1}{r_n a_n} + \frac{\varepsilon_n}{a_n} + \frac{1}{\sqrt{r_n a_n}} \right), \end{aligned}$$

a contradiction, since $r_n a_n \rightarrow \infty$ and $\varepsilon_n / a_n = c / s_n \rightarrow 0$.

We now return to the proof of the necessity of (5). Define the continuous 2π -periodic function F_n by $F_n(t_{kn}) = (-1)^k$ ($1 \leq k \leq n$), F_n is linear in between, constant in $[0, t_{1n}]$, $[t_{nn}, \pi]$, $F_n(t) = F_n(-t)$ ($-\pi \leq t \leq 0$), and $F_n(t + 2\pi) = F_n(t)$ ($-\infty < t < \infty$). By (15) $\omega(F_n, h) \leq cnh$, hence $E_n^T(F_n) \leq c_1$. Set $f_n(x) = F_n(\arccos x)$. Then by assumption for any $\varepsilon > 0$ there exist $p_n \in \Pi_{[(1+\varepsilon)n]}$ such that $p_n(x_{kn}) = f_n(x_{kn}) = (-1)^k$ ($1 \leq k \leq n$) and

$$\|f_n - p_n\| \leq c_\varepsilon E_{[(1+\varepsilon)n]}(f_n) = c_\varepsilon E_{[(1+\varepsilon)n]}^T(F_n) \leq \tilde{c}_\varepsilon.$$

Thus $\|p_n\| \leq c_\varepsilon^*$ ($\deg p_n = [(1 + \varepsilon)n]$); hence by Lemma 3

$$\overline{\lim}_{n \rightarrow \infty} \frac{N_n(I_n)}{[(1 + \varepsilon)n]|I_n|} \leq \overline{\lim}_{n \rightarrow \infty} \frac{Q(I_n)}{[(1 + \varepsilon)n]|I_n|} \leq \frac{1}{\pi}.$$

Since $\varepsilon > 0$ is arbitrary, we can put $\varepsilon = 0$ here.

Using the same arguments, we could have proved the following, slightly more general theorem:

THEOREM A. *For every $f(x) \in C[-1, 1]$, $\varepsilon > 0$, and $d \geq 1$ there exists a sequence of polynomials $q_n(x) \in \Pi_{[dn(1 + \varepsilon)]}$ such that (2) and*

$$\|f(x) - q_n(x)\| = O(E_{[dn(1 + \varepsilon)]}(f))$$

hold, if and only if

$$\overline{\lim}_{n \rightarrow \infty} \frac{N_n(I_n)}{n|I_n|} \leq \frac{d}{\pi} \quad \text{whenever} \quad \lim_{n \rightarrow \infty} n|I_n| = \infty$$

and (6) holds.

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